

Singular cosmological evolution using canonical and ghost scalar fields

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ABSTRACT: We demonstrate that finite time singularities of Type IV can be consistently incorporated in the Universe's cosmological evolution, either appearing in the inflationary era, or in the late-time regime. While using only one scalar field instabilities can in principle occur at the time of the phantom-divide crossing, when two fields are involved we are able to avoid such instabilities. Additionally, the two-field scalar-tensor theories prove to be able to offer a plethora of possible viable cosmological scenarios, at which various types of cosmological singularities can be realized. Amongst others, it is possible to describe inflation with the appearance of a Type IV singularity, and phantom late-time acceleration which ends in a Big Rip. Finally, for completeness, we also present the Type IV realization in the context of suitably reconstructed $F(R)$ gravity.

KEYWORDS: Singular evolution, quintessence, phantom, ghost, scalar field, two fields, $F(R)$ gravity

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1 Introduction

The observational data concerning the effective equation-of-state (EoS) parameter of dark energy w_{eff} , slightly indicate that in the recent cosmological past the EoS parameter might have crossed the phantom divide and lies between quintessence and phantom regimes [1–4], and therefore the cosmological evolution of the present Universe might be described by an effective phantom phase. There might also be the possibility that although the current Universe may be of quintessential type, the phantom era may occur in the near future. For an incomplete list of articles discussing the possibility that our Universe is described by an effective late-time phantom era see [1–16] and references therein. On the other hand, the inflationary era is also an accelerating phase of the Universe, which is not up to date excluded to be of quintessence or phantom type. One of the biggest problems in this type of phenomenology, is to fully understand the relation and the connection of early-time inflation with late-time acceleration, and in particular how to consistently describe these accelerating eras using the same theoretical framework. The origin of the problem itself might be the incomplete understanding we have, up to date, of dark energy and how the transition from deceleration to the dark energy acceleration epoch is eventually achieved.

As was demonstrated in [1], a promising theoretical unified description of both early-time phantom inflation and late-time phantom acceleration is achieved by using canonical and non-canonical scalar fields. Using this generalized multiple-scalar-field scalar-tensor theory, with generalized scalar-field-dependent kinetic functions in front of the kinetic terms and with a generalized scalar potential, a consistent theoretical framework can be provided for these theories [17]. This can be done either by using two scalar fields, or by using even a single scalar field. However, in the latter case there is a theoretical obstacle connected with the transition time at which the EoS evolves from non-phantom to phantom and vice-versa. In particular, the single scalar field theoretical description of cosmological dynamics always comes with the cost of possible instabilities at the cosmic time that the phantom divide is crossed, since at exactly this point the dynamical system that describes the evolution of the Universe can be unstable towards linear perturbations [18], rendering the resulting scalar-tensor theory unstable. Hence, the use of two scalar fields is necessary in order to resolve these issues in a concrete way [1, 19].

Definitely, the single-field scalar-tensor theory can in some cases prove valuable, since it offers the opportunity to describe exotic cosmological scenarios, which were not possible to be realized in the context of standard Einstein-Hilbert general relativity. In a recent study [20] we addressed the issue of having an inflationary era in the presence of finite-time singularities, with special emphasis in the Type IV singularity. Particularly, as we explicitly have shown, it is possible to have viable phenomenology even in the presence of Type IV singularities, and in some cases compatible with observational data coming from Planck [21] and BICEP2 [22] collaborations. These finite-time singularities can be of various types and were classified in [23], with regards to their relevance in the cosmological evolution. These types vary, with the main criterion that classifies them being the fact that certain observable quantities may become infinite at the time these occur.

In principle, a singularity in a cosmological context is an unwanted feature of cosmological evolution. Space-time singularities were studied long ago by Hawking and Penrose [24], and they are defined as isolated points in space-time at which the space-time is geodesically incomplete. This feature practically means that there exist null and time-like geodesics, which cannot be continuously extended to arbitrary values of their parameters. The most characteristic example of this type of singularity is the initial singularity of our Universe, which naturally appears in inflationary theories [25–28]. It is worthy to mention however that there exist theories which are free of this initial singularity, such as the bouncing Universe scenarios [29–50]. Apart from these severe singularities however, there exist singularities at which some observable quantities may become infinite, without this implying geodesic incompleteness. These are called sudden singularities and were extensively studied by Barrow [51] (see also [52–74] for some later studies). In these singularities, the scale factor remains finite at the singular point, and therefore these are not singularities of crushing type, such as the Big Rip [10–12, 23, 75–92].

In our recent study [20], the focus was on the existence of a non-crushing type of singularity, namely the Type IV. Our theoretical description of the cosmological evolution involved a scalar-tensor theory with a single scalar field. However, when someone uses a single scalar field, the scalar-tensor description might be unstable at the cosmic time

at which the phantom divide is crossed. This was not the case in our previous article, but this is a clear possibility when someone deals with single-field scalar-tensor theories. Therefore, the purpose of the present article is two-fold. Firstly, we aim to highlight the problem of instability in the single-field scalar-tensor cosmologies, in the presence of finite time singularities. Secondly, we are interested to formally address the crossing from the non-phantom to the phantom era consistently. The latter issue is definitely connected to the first one. We shall exemplify the problem by using a single-field scalar-tensor theory to produce a variant of a well known potential from inflationary cosmology, namely the hilltop potential [93–100], in which we consistently incorporate a Type IV singularity. As we show, the solution is unstable at the time the EoS crosses the phantom divide, and consequently the two-scalar-field cosmological description is rendered compelling. Having done this, we shall study illustrative examples of two-field scalar-tensor cosmologies, in which finite-time singularities exist. More importantly, we shall explicitly demonstrate that it is possible to unify phantom (or non-phantom) inflation, with phantom (or non-phantom) late-time acceleration. Finally, for completeness, we shall investigate which $F(R)$ gravity can generate a cosmological evolution corresponding to the hilltop scalar-tensor model, near the Type IV singularity. Interestingly enough, if the Type IV singularity is assumed to occur at late times, the resulting $F(R)$ gravity is of the form $R + \frac{A}{R}$, a well known model that can consistently describe late-time acceleration [101].

This paper is organized as follows: In section 2 we show that the single-field scalar-tensor theory may lead to instabilities at the phantom-divide crossing, which are removed with the use of a second field. In section 3 we proceed to the investigation of various specific cosmological evolutions in the framework of two-field scalar-tensor theory, which prove the capabilities of the construction. In particular, in subsection 3.1 we reconstruct scenarios which exhibit Type II or Type IV singularities, in subsection 3.2 we analyze the slow-roll inflationary realization in such theories, in 3.3 we reconstruct a universe with a singular inflationary phase that results in a dark energy epoch which ends in a Big Rip, in 3.4 we present the two-field Barrow’s model, while in subsection 3.5 we extend our investigation in the case of multiple scalar fields. In section 4, for completeness, we also present the Type IV realization in the context of suitably reconstructed $F(R)$ gravity. Finally, in section 5 we summarize our results.

2 Scalar-tensor gravity: general analysis

In this section we shall examine in detail how the finite-time singularities can be consistently incorporated in generic scalar-tensor theories containing two scalar fields. In addition, we will demonstrate why the need for two scalars is in some cases compelling. Using well known reconstruction methods [1, 102], for a given cosmological evolution we shall provide the details of the generic scalar-tensor models that can consistently incorporate finite-time singularities, and then in the next section we will explicitly demonstrate our results using some illustrative examples. We mention to the reader that apart from Refs. [1, 102], many reconstruction techniques had been developed earlier, and most of these studies are very well reviewed in Ref. [103].

In this work we consider a spatially flat Friedmann-Robertson-Walker (FRW) metric of the form

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \quad (2.1)$$

with $a(t)$ the scale factor. In this case, the effective energy density ρ_{eff} and the effective pressure p_{eff} are defined as

$$\rho_{\text{eff}} \equiv \frac{3}{\kappa^2} H^2, \quad p_{\text{eff}} \equiv -\frac{1}{\kappa^2} (2\dot{H} + 3H^2), \quad (2.2)$$

where $H = \dot{a}/a$ denotes the Hubble parameter and dots indicate differentiation with respect to the cosmic time t .

Let us first recall the finite-time cosmological singularities classification. For details on this subject the reader is referred to [23, 104–106]. According to [23, 106], the finite-time future singularities are classified in the following way:

- Type I (“Big Rip”) : As $t \rightarrow t_s$, the scale factor a , the effective energy density ρ_{eff} , and the effective pressure p_{eff} diverge, namely $a \rightarrow \infty$, $\rho_{\text{eff}} \rightarrow \infty$, and $|p_{\text{eff}}| \rightarrow \infty$. For a presentation of this type of singularity, see Ref. [75] and also Ref. [23].
- Type II (“sudden”): As $t \rightarrow t_s$, both the scale factor and the effective energy density are finite, that is $a \rightarrow a_s$, $\rho_{\text{eff}} \rightarrow \rho_s$, but the effective pressure diverges, namely $|p_{\text{eff}}| \rightarrow \infty$ [51–73].
- Type III: As $t \rightarrow t_s$, the scale factor is finite, $a \rightarrow a_s$ but the effective energy density and the effective pressure diverge, $\rho_{\text{eff}} \rightarrow \infty$, $|p_{\text{eff}}| \rightarrow \infty$ [23].
- Type IV: As $t \rightarrow t_s$, the scale factor, the effective energy density, and the effective pressure are finite, namely $a \rightarrow a_s$, $\rho_{\text{eff}} \rightarrow \rho_s$, $|p_{\text{eff}}| \rightarrow p_s$, but the Hubble’s rate higher derivatives diverge, that is $H \equiv \dot{a}/a$ [23].

Finally, note that in the aforementioned list of singularities, once should add the standard case of recollapse due to a positive spatial curvature ending in the “Big Crunch” [107].

In a previous work [20] we studied how singular inflation can be properly accommodated to a power-law model of inflation, studied by Barrow in [74]. Using the scalar field reconstruction method [1, 102], we found which scalar-tensor gravities can lead to a successful description of such inflationary cosmology. However, the conventions we used in the previous article lead to a stable scalar-tensor solution, which had an effective equation of state characteristic of quintessential acceleration, as long as the power-law inflation model is considered. Nevertheless, the stability of the scalar-tensor solutions is not ensured in general, and instabilities might occur during the reconstruction process. Therefore, in this section we explicitly demonstrate how instabilities can indeed appear, by using two phenomenologically appealing inflationary models. As we evince, the simplest way to avoid the instabilities is to use two (or more) scalar fields, with one of them being ghost and the other scalar field being canonical.

In a first subsection we aim to show how the Type IV singularities can occur in a slightly deformed version of a very well known viable inflationary model, namely the hilltop inflationary potential [93–100]. Thus, in a next subsection we will be able to provide a consistent framework consisting of two scalar fields for the aforementioned inflationary cosmologies.

2.1 Singular evolution with hilltop-like potentials: Description with a single scalar and the singularity appearance

We start our analysis with the hilltop potentials, and we demonstrate how a Type IV singularity can be consistently incorporated in the theoretical framework of these models by using a very general reconstruction scheme for scalar-tensor theories. Before getting into the main analysis, it is worthy to give a brief description of the scalar reconstruction method we shall use. For a detail presentation on this issue the reader is referred to [1, 102].

We consider the following non-canonical scalar-tensor action, which describes a single scalar field:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - V(\phi) + L_{\text{matter}} \right\}, \quad (2.3)$$

with the function $\omega(\phi)$ being the kinetic function and $V(\phi)$ the scalar potential. The kinetic-term function $\omega(\phi)$ appearing in (2.3) is irrelevant, since it can be consistently absorbed by appropriately redefining the scalar field ϕ as

$$\varphi \equiv \int^\phi d\phi \sqrt{\omega(\phi)}, \quad (2.4)$$

where it is assumed that $\omega(\phi) > 0$. Hence, the kinetic term of the scalar field appearing in action (2.3) can be written as

$$- \omega(\phi) \partial_\mu \phi \partial^\mu \phi = - \partial_\mu \varphi \partial^\mu \varphi. \quad (2.5)$$

On the other hand, if $\omega(\phi) < 0$ then the scalar field becomes ghost, and it can describe the phantom dark energy. In this case, instead of (2.4) one makes the redefinition

$$\varphi \equiv \int^\phi d\phi \sqrt{-\omega(\phi)}, \quad (2.6)$$

and thus, instead of (2.5), one acquires the expression

$$- \omega(\phi) \partial_\mu \phi \partial^\mu \phi = \partial_\mu \varphi \partial^\mu \varphi. \quad (2.7)$$

When the scalar field is a ghost field the energy density corresponding to the classical theory becomes unbounded from below. At the quantum level the energy could be rendered bounded from below, however at the significant cost of the appearance of a negative norms [108, 109].

In both canonical and phantom cases, in the case of FRW geometry the energy density and pressure of the scalar field write as

$$\rho = \frac{1}{2} \omega(\phi) \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \omega(\phi) \dot{\phi}^2 - V(\phi). \quad (2.8)$$

Having these at hand, we can use the usual Friedmann equations in the absence of the matter sector, in order to promptly express the scalar potential $V(\phi)$ and the kinetic term $\omega(\phi)$ in terms of the Hubble rate and its first derivative as

$$\omega(\phi)\dot{\phi}^2 = -\frac{2}{\kappa^2}\dot{H}, \quad V(\phi) = \frac{1}{\kappa^2} \left(3H^2 + \dot{H} \right). \quad (2.9)$$

The scalar-reconstruction method is based on the assumption that the kinetic term $\omega(\phi)$ and the scalar potential $V(\phi)$, can be written in terms of a single function $f(\phi)$ in the following way:

$$\omega(\phi) = -\frac{2}{\kappa^2}f'(\phi), \quad V(\phi) = \frac{1}{\kappa^2} \left[3f(\phi)^2 + f'(\phi) \right]. \quad (2.10)$$

Therefore, the equations (2.9) become

$$\phi = t, \quad H = f(t). \quad (2.11)$$

Finally, by varying the action in terms of the scalar field we obtain its evolution equation as

$$0 = \omega(\phi)\ddot{\phi} + \frac{1}{2}\omega'(\phi)\dot{\phi}^2 + 3H\omega(\phi)\dot{\phi} + V'(\phi), \quad (2.12)$$

which is also satisfied by the solution (2.11).

In order to proceed, we need to make a choice for the scalar potential. In the case of a canonical scalar field, the hilltop inflation potential [93–100] has been proved to be consistent with the Planck data [21], and it is approximately given by

$$V(\varphi) \simeq \mathcal{A} - \mathcal{B}\varphi^p + \dots, \quad (2.13)$$

with the ellipsis denoting higher-order terms which are negligible during inflation. When the parameters are chosen appropriately, and amongst others $p = 2$, the implications of this inflationary mode are in 95% agreement with the Planck data [21]. Hence, we prefer to consider a case close to this phenomenologically well-behaved case, and we assume $p \simeq 2 - \epsilon$, with ϵ an infinitesimally small number $|\epsilon| \ll 1$.

Let us now start the main part of the analysis of this subsection, which is to examine whether there exist finite-time singularities in the hilltop potential scenario (2.13). As we shall demonstrate, it is possible to connect a Type IV and a Type I singularity with potentials having a functional resemblance to the hilltop potential. However, it is impossible to fully satisfy the constraints posed by Planck in these hilltop-like potentials, and therefore we should stress that the following considerations are excluded by observations. Nevertheless, for the moment we are interested in demonstrating that instabilities will appear, and we do not consider consistent confrontation with observations. One could construct more complicated single-field scenarios, in agreement with observations, and with the appearance of singularities, however this would lie beyond our scope, which is to show in a simple single-field model that singularities appear.

In order to quantify the appearance of a singularity we parameterize the Hubble rate as

$$H(t) = c_0 + b_0 (t_s - t)^\alpha, \quad (2.14)$$

where c_0, b_0 are arbitrary positive constants, and later on we shall determine which values of α are allowed. Clearly, according to the value of α , one could have the appearance of the following singularities:

- $\alpha < -1$ corresponds to Type I singularity.
- $-1 < \alpha < 0$ corresponds to Type III singularity.
- $0 < \alpha < 1$ corresponds to Type II singularity.
- $\alpha > 1$ corresponds to Type IV singularity.

Inserting the above Hubble ansatz into (2.10), we may directly find the scalar potential and the kinetic term as

$$\omega(\phi) = \frac{2b_0\alpha(t_s - \phi)^{-1+\alpha}}{\kappa^2}, \quad (2.15)$$

$$V(\phi) = \frac{3c_0^2}{\kappa^2} - \frac{b_0\alpha(t_s - \phi)^{-1+\alpha}}{\kappa^2} + \frac{6b_0c_0(t_s - \phi)^\alpha}{\kappa^2} + \frac{3b_0^2(t_s - \phi)^{2\alpha}}{\kappa^2}. \quad (2.16)$$

Additionally, imposing the transformation (2.4) we obtain

$$\varphi = -\frac{2\sqrt{2\alpha b_0}}{\kappa(\alpha+1)}(t_s - \phi)^{\frac{\alpha+1}{2}}, \quad (2.17)$$

and therefore the scalar potential (2.16) becomes

$$V(\varphi) \sim \frac{3c_0^2}{\kappa^2} - \frac{b_0\alpha\mathcal{A}_1^{\alpha-1}}{\kappa^2}\varphi^{\frac{2(-1+\alpha)}{\alpha+1}} + \frac{6b_0c_0\mathcal{A}_1^\alpha}{\kappa^2}\varphi^{\frac{2\alpha}{\alpha+1}} + \frac{3b_0^2\mathcal{A}_1^{2\alpha}}{\kappa^2}\varphi^{\frac{4\alpha}{\alpha+1}}, \quad (2.18)$$

with

$$\mathcal{A}_1 = \left[-\frac{2\sqrt{2\alpha b_0}}{\kappa(\alpha+1)} \right]^{\frac{2}{\alpha+1}}. \quad (2.19)$$

Since we are interested in small values of φ , which according to (2.11) corresponds to the case where $t_s - \phi \rightarrow 0$, we may neglect the higher-order terms in the potential (2.18), and approximate it as

$$V(\varphi) \approx \frac{3c_0^2}{\kappa^2} - \frac{b_0\alpha\mathcal{A}_1^{-1+\alpha}}{\kappa^2}\varphi^{\frac{2(\alpha-1)}{\alpha+1}}. \quad (2.20)$$

In the case $p = 2 - \epsilon$ this potential coincides with (2.13) under the identification

$$\alpha = \frac{2 + \epsilon}{\epsilon}, \quad (2.21)$$

which is a very large positive number as $\epsilon \rightarrow 0$, if $\epsilon > 0$. Hence, in this case the system develops a Type IV singularity. On the other hand, when $p > 2$ the system develops a Type I singularity, since in this case comparing (2.13) with (2.20) we obtain

$$\alpha = \frac{p+2}{2-p}, \quad (2.22)$$

which is clearly negative for $p > 2$ (i.e. $\epsilon < 0$).

Before proceeding, let us make a comment on the above reconstruction procedure. Actually, this procedure is based on the Hamilton-Jacobi-like approach independently introduced in [110, 111], however now the difference is that instead of solving the resulting first-order nonlinear differential equation for $H(\phi)$ (or equivalently $H(t)$) we impose it at will, and we calculate the potential $V(\phi)$ required for the given dynamical evolution. Strictly speaking, one should examine whether there are other solutions corresponding to the same potential, but for the purposes of this work it is adequate to extract one of them.

We now calculate the effective equation of state

$$w_{\text{eff}} = \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2}. \quad (2.23)$$

Using the Hubble parameter (2.14) we obtain

$$w_{\text{eff}} = -1 + \frac{2b_0(-t + t_s)^{-1+\alpha}\alpha}{3[c_0 + b_0(-t + t_s)^\alpha]^2}. \quad (2.24)$$

In one of the interesting cases at hand, namely when $\alpha \gg 1$, which corresponds to the Type IV singularity, the effective equation-of-state parameter becomes exactly equal to $w_{\text{eff}} = -1$ at the time $t = t_s$. This case is problematic for the following reason: As one can see, the main source of the problem originates from the fact that according to (2.14) we obtain $\dot{H} = 0$ at $t = t_s$. This can potentially introduce a very large instability when crossing the phantom divide $w_{\text{eff}} = -1$. In order to show this explicitly we introduce the variables X_ϕ and Y , defined as

$$X_\phi = \dot{\phi}, \quad Y = \frac{f(\phi)}{H}. \quad (2.25)$$

In practice, the variable Y quantifies the deviation from the solution of the reconstructed scalar-tensor theory, given in (2.11). Using these new variables, the Friedmann equations (2.9) and the field equation (2.12) can be rewritten as

$$\frac{dX_\phi}{dN} = \frac{f''(\phi)(X_\phi^2 - 1)}{2f'(\phi)H} - 3(X_\phi - Y), \quad \frac{dY}{dN} = \frac{f'(\phi)(1 - X_\phi Y)X_\phi}{H^2}, \quad (2.26)$$

where N is the e -folding number. Hence, the scalar-tensor reconstruction solution of (2.11) corresponds to the following values for the new variables:

$$X_\phi = 1, \quad Y = 1, \quad (2.27)$$

which correspond exactly to the basic critical point of system (2.26). As usual, in order to examine when this solution is stable we linearly perturb it around this critical point as [112–114]

$$X_\phi = 1 + \delta X_\phi, \quad Y = 1 + \delta Y, \quad (2.28)$$

and thus the dynamical system (2.26) can be re-written as

$$\frac{d}{dN} \begin{pmatrix} \delta X_\phi \\ \delta Y \end{pmatrix} = \begin{pmatrix} -\frac{\ddot{H}}{H\dot{H}} - 3 & 3 \\ -\frac{\ddot{H}}{H^2} & -\frac{\dot{H}}{H^2} \end{pmatrix} \begin{pmatrix} \delta X_\phi \\ \delta Y \end{pmatrix}. \quad (2.29)$$

Hence, the stability of the dynamical system (2.29) is ensured if the eigenvalues of the involved 2×2 matrix are negative. These eigenvalues in general read as

$$\begin{aligned}
M_+ &= \frac{1}{2} \left[- \left(\frac{H''(t)}{H'(t)H(t)} + \frac{H'(t)}{H(t)^2} + 3 \right) \right. \\
&\quad \left. + \sqrt{\left(\frac{H''(t)}{H'(t)H(t)} + \frac{H'(t)}{H(t)^2} + 3 \right)^2 - \frac{4H''(t)}{H(t)^3} - \frac{12H'(t)}{H(t)^2}} \right], \\
M_- &= \frac{1}{2} \left[- \left(\frac{H''(t)}{H'(t)H(t)} + \frac{H'(t)}{H(t)^2} + 3 \right) \right. \\
&\quad \left. - \sqrt{\left(\frac{H''(t)}{H'(t)H(t)} + \frac{H'(t)}{H(t)^2} + 3 \right)^2 - \frac{4H''(t)}{H(t)^3} - \frac{12H'(t)}{H(t)^2}} \right], \tag{2.30}
\end{aligned}$$

and thus for the Hubble rate (2.14), at the time $t \simeq t_s$, they become

$$M_+ = 0, \quad M_- = \frac{1}{2} \left[-6 + \frac{2(-1 + \alpha)}{(-t + t_s)c_0} \right]. \tag{2.31}$$

Since $M_- > 0$ when $t < t_s$ and $M_- < 0$ when $t > t_s$, the dynamical system is unstable at the transition point $t = t_s$, which corresponds to the phantom-divide transition point. This result is valid for any value of the parameter α . In addition, at the transition point the eigenvalue M_- is infinite. This instability was first observed in [18, 115]. In order to evade this kind of problems in scalar-tensor cosmologies, the reconstruction method must be enriched with the presence of two scalars. As we show in the next subsection, the instability does not occur in such a case.

2.2 Singular evolution with hilltop-like potentials: Description with two scalar fields

As we demonstrated in the previous subsection, when we consider the transition from the non-phantom phase to the phantom one, in the context of scalar-tensor gravity with one scalar field, there appears an infinite instability at the transition point [1, 18, 102, 115]. However, this instability can easily be removed by considering scalar-tensor theories with two scalar fields.

Consider the following two-field scalar-tensor gravity, with action:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \eta(\chi) \partial_\mu \chi \partial^\mu \chi - V(\phi, \chi) \right\}. \tag{2.32}$$

As in the single scalar case, the function $\omega(\phi)$ is the kinetic function for the scalar field ϕ , and accordingly $\eta(\chi)$ is the kinetic function for the scalar field χ . When the functions $\omega(\phi)$ or $\eta(\chi)$ are negative the corresponding scalar field becomes a ghost field. However, since such a field would lead to inconsistencies at the quantum level [108], one expects it to arise through an effective description of a non-phantom fundamental theory [116].

Assuming that the two scalar fields ϕ and χ depend only on the cosmic-time coordinate t , and also that the spacetime is described by a flat FRW metric of the form (2.1), we easily

write the Friedmann equations in the form

$$\omega(\phi)\dot{\phi}^2 + \eta(\chi)\dot{\chi}^2 = -\frac{2}{\kappa^2}\dot{H}, \quad (2.33)$$

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left(3H^2 + \dot{H} \right). \quad (2.34)$$

Following the procedure of the previous subsection, and generalizing it in the case of two fields, we impose the parametrization

$$\omega(t) + \eta(t) = -\frac{2}{\kappa^2}f'(t), \quad V(t) = \frac{1}{\kappa^2} [3f(t)^2 + f'(t)], \quad (2.35)$$

which is consistent with an explicit solution for (2.33) and (2.34) of the form

$$\phi = \chi = t, \quad H = f(t). \quad (2.36)$$

For a detailed account on this reconstruction method see [1].

The kinetic functions $\omega(\phi)$ and $\eta(\chi)$ can be chosen in such a way that $\omega(\phi)$ is always positive and $\eta(\chi)$ is always negative, hence one field is canonical and the other one is ghost. This is exactly the realization of the quintom scenario [4]. In particular, a convenient choice for the kinetic functions is

$$\begin{aligned} \omega(\phi) &= -\frac{2}{\kappa^2} \left\{ f'(\phi) - \sqrt{\alpha_1(\phi)^2 + f'(\phi)^2} \right\} > 0, \\ \eta(\chi) &= -\frac{2}{\kappa^2} \sqrt{\alpha_1(\chi)^2 + f'(\chi)^2} < 0, \end{aligned} \quad (2.37)$$

where $\alpha_1(x)$ is an arbitrary function of its argument. We now define a new function $\tilde{f}(\phi, \chi)$ through

$$\tilde{f}(\phi, \chi) \equiv -\frac{\kappa^2}{2} \left[\int d\phi \omega(\phi) + \int d\chi \eta(\chi) \right], \quad (2.38)$$

which has the characteristic property

$$\tilde{f}(t) = f(t), \quad (2.39)$$

and therefore the constants of integration are fixed in (2.38). Assuming that $V(\phi, \chi)$ is given in terms of the function $\tilde{f}(\phi, \chi)$ as

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left[3\tilde{f}(\phi, \chi)^2 + \frac{\partial \tilde{f}(\phi, \chi)}{\partial \phi} + \frac{\partial \tilde{f}(\phi, \chi)}{\partial \chi} \right], \quad (2.40)$$

we find that, in addition to the Friedmann equations given in (2.33) and (2.34), the following two-scalar field equations are also satisfied:

$$0 = \omega(\phi)\ddot{\phi} + \frac{1}{2}\omega'(\phi)\dot{\phi}^2 + 3H\omega(\phi)\dot{\phi} + \frac{\partial \tilde{V}(\phi, \chi)}{\partial \phi}, \quad (2.41)$$

$$0 = \eta(\chi)\ddot{\chi} + \frac{1}{2}\eta'(\chi)\dot{\chi}^2 + 3H\eta(\chi)\dot{\chi} + \frac{\partial \tilde{V}(\phi, \chi)}{\partial \chi}. \quad (2.42)$$

In summary, upon using the above kinetic functions $\omega(\phi)$, $\eta(\chi)$ and the scalar potential $V(\phi, \chi)$, we have a two-scalar field scalar-tensor model with cosmological evolution of the form given in (2.36).

Let us now extend the analysis of the previous subsection, and proceed to the reconstruction of a two-field scalar-tensor gravity generating the cosmology described by the scale factor (2.14). As we discussed, this class of cosmological evolution is related to a single-field scalar-tensor theory, which generates a hilltop-like scalar potential of the form (2.13) for the corresponding canonical scalar field. Moreover, it exhibits specific finite-time singularities, according to the values of α given in the list below (2.14).

In order to proceed, we need to impose an ansatz for the arbitrary function $\alpha_1(x)$ appeared in (2.37). We choose it as

$$\alpha_1(x) = \sqrt{b_0 \alpha (-x + t_s)^{\alpha-1} - H'(x)^2}, \quad (2.43)$$

and therefore (2.37) give

$$\begin{aligned} \omega(\phi) &= \frac{2 \left[b_0 \alpha (t_s - \phi)^{-1+\alpha} + \sqrt{b_0 \alpha (t_s - \phi)^{-1+\alpha}} \right]}{\kappa^2}, \\ \eta(\chi) &= -\frac{2 \sqrt{b_0 \alpha (t_s - \chi)^{-1+\alpha}}}{\kappa^2}. \end{aligned} \quad (2.44)$$

In order to avoid inconsistencies we assume that α has the form

$$\alpha = \frac{n}{2m+1}, \quad (2.45)$$

and therefore when n is an odd integer, the kinetic function $\omega(\phi)$ is positive, while $\eta(\chi)$ is negative. The corresponding function $\tilde{f}(\phi, \chi)$ from (2.38) writes as

$$\tilde{f}(\phi, \chi) = -b_0 (t_s - \phi)^\alpha - \frac{2\sqrt{b_0 \alpha} (t_s - \phi)^{\frac{\alpha+1}{2}}}{(\alpha+1)} - \frac{2\sqrt{b_0 \alpha} (t_s - \chi)^{\frac{\alpha+1}{2}}}{(\alpha+1)}, \quad (2.46)$$

and thus the scalar potential $V(\phi, \chi)$ in (2.40) becomes

$$\begin{aligned} V(\phi, \chi) &= \frac{b_0 \alpha (t_s - \phi)^{\alpha-1}}{\kappa^2} + \frac{2\sqrt{b_0 \alpha} (t_s - \phi)^{\frac{\alpha-1}{2}} + \sqrt{b_0 \alpha} (\alpha-1) (t_s - \phi)^{\frac{\alpha-1}{2}}}{(1+\alpha)\kappa^2} \\ &+ \frac{2\sqrt{b_0 \alpha} (t_s - \chi)^{\frac{\alpha-1}{2}}}{(1+\alpha)\kappa^2} - \frac{\sqrt{b_0 \alpha} (\alpha-1) (t_s - \chi)^{\alpha-1}}{(1+\alpha)\kappa^2} \\ &+ \frac{3}{\kappa^2} \left\{ b_0 (t_s - \phi)^\alpha + \frac{2\sqrt{b_0 \alpha} (t_s - \phi)^{\frac{\alpha+1}{2}}}{(\alpha+1)} + \frac{2\sqrt{b_0 \alpha} (t_s - \chi)^{\frac{\alpha+1}{2}}}{(\alpha+1)} \right\}^2. \end{aligned} \quad (2.47)$$

For the above two-field scalar-tensor theory, the solution (2.36) is stable as we will demonstrate in detail. Indeed, we introduce the quantities X_ϕ , X_χ and \bar{Y} through

$$X_\phi = \dot{\phi}, \quad X_\chi = \dot{\chi}, \quad \bar{Y} = \frac{\tilde{f}(\phi, \chi)}{H}, \quad (2.48)$$

and thus the Friedmann equations (2.33), combined with (2.41) and (2.42), can be expressed as

$$\begin{aligned}\frac{dX_\phi}{dN} &= \frac{\omega'(\phi)(X_\phi^2 - 1)}{2\omega(\phi)H} - 3(X_\phi - \bar{Y}), \\ \frac{dX_\chi}{dN} &= \frac{\eta'(\chi)(X_\chi^2 - 1)}{2\eta(\chi)H} - 3(X_\chi - \bar{Y}), \\ \frac{d\bar{Y}}{dN} &= \frac{3X_\phi X_\chi(1 - \bar{Y}^2)}{X_\phi + X_\chi} + \frac{\dot{H}}{H^2} \frac{X_\phi X_\chi + 1 - \bar{Y}(X_\phi + X_\chi)}{X_\phi + X_\chi}.\end{aligned}\quad (2.49)$$

The solution appearing in (2.36) corresponds to the following values of the variables X_ϕ , X_χ and \bar{Y} :

$$X_\phi = 1, \quad X_\chi = 1, \quad \bar{Y} = 1, \quad (2.50)$$

which correspond exactly to the basic critical point of system (2.49). As we did in the previous subsection, in order to examine when this solution is stable we linearly perturb it around this critical point as

$$X_\phi = 1 + \delta X_\phi, \quad X_\chi = 1 + \delta X_\chi, \quad \bar{Y} = 1 + \delta \bar{Y}, \quad (2.51)$$

obtaining the perturbative dynamical system

$$\frac{d}{dN} \begin{pmatrix} \delta X_\phi \\ \delta X_\chi \\ \delta \bar{Y} \end{pmatrix} = \begin{pmatrix} -\frac{\omega'(\phi)}{H\omega(\phi)} - 3 & 0 & 3 \\ 0 & -\frac{\eta'(\chi)}{H\eta(\chi)} & 3 \\ 0 & 0 & -3 - \frac{\dot{H}}{H^2} \end{pmatrix} \begin{pmatrix} \delta X_\phi \\ \delta X_\chi \\ \delta \bar{Y} \end{pmatrix}. \quad (2.52)$$

The eigenvalues of the matrix appearing in the dynamical system (2.52) are found to be

$$M_\phi = -\frac{\omega'(\phi)}{H\omega(\phi)} - 3, \quad M_\chi = -\frac{\eta'(\chi)}{H\eta(\chi)} - 3, \quad M_{\bar{Y}} = -3 - \frac{\dot{H}}{H^2}. \quad (2.53)$$

Hence, in the case where the Hubble rate is given by (2.2), these eigenvalues become

$$\begin{aligned}M_\phi &= -3 - \frac{(-1 + \alpha) \left(1 + 2\sqrt{b_0\alpha(t_s - \phi)^{-1+\alpha}}\right) \sqrt{b_0\alpha(t_s - \phi)^{-1+\alpha}}}{2c_0 \left[b_0\alpha(t_s - \phi)^\alpha + (t_s - \phi)\sqrt{b_0\alpha(t_s - \phi)^{-1+\alpha}}\right]}, \\ M_\chi &= -3 - \frac{-1 + \alpha}{2c_0(t_s - \chi)}, \\ M_{\bar{Y}} &= -3,\end{aligned}\quad (2.54)$$

which are clearly negative for all values of the cosmic time t . Therefore, we deduce that the solution (2.36), with the Hubble rate given in (2.2), is stable. Hence, as we have already mentioned, the presence of two scalar fields is adequate to solve the instability problem of the one-field theory. Having the stable solution (2.36) at hand, we establish the fact that a Type IV singularity can underlie the cosmology generated by the Hubble rate (2.2). In particular, one possible description can be realized by a two-field generic scalar-tensor

theory with kinetic functions given in (2.44) and with the scalar potential appearing in (2.47).

Finally, it is worthy to mention that the solution (2.36) is only one particular solution of the dynamical system (2.33) with (2.41) and (2.42). In principle, there might exist other solutions, however the stability of (2.36) which renders it an attractor, ensures that this solution will be the asymptotic limit of a large class of solutions.

2.3 Singular versus non-singular evolution

Let us close this section by making some comments on the possible removal of the singularities discussed above. In the literature there are mechanisms that could remove some singularities of a given cosmological scenario. Two of the known such mechanisms are the inclusion of quantum effects, or the suitable slight modification of the action (for instance changing the $F(R)$ or the potential form [117]).

Indeed, singularities can become milder through the incorporation of quantum effects in a perturbative way, without changing the action. However this is not the case for the singularities analyzed in the previous subsections (in general even the quantum description of the phantom regime is ambiguous [108, 109]).

Nevertheless, one could regularize and remove some of these singularities by modifying the action, i.e. by slightly changing the potential [117]. However, following this approach is not the scope of our analysis. In particular, we would like to mention that in principle the potential of a model is expected in general to be determined by the fundamental theory itself, thus in general one cannot modify it at will in a suitable way that will make the singularities to be absent. Hence, since any potential could arise from a fundamental theory, even if it is of special form, including the ones considered in this work, it is interesting and necessary to investigate their cosmological implications and in particular the realization of singularities, without entering the discussion to modify it, i.e. modifying the theory. Indeed, a theory admitting weak singularities has to be special in some respect (for instance note that there are not such singularities in the vacuum general relativity case), however this could still be the case in Nature, deserving further investigation.

Additionally, note that there could still be cases where weak singularities cannot be removed even after a slight modification of the action. For instance, as it was shown in [118], the singularities arising in $F(R)$ gravity in the case where $F''(R_0) = 0$ for some $R = R_0$, may not be removed through the procedure of [117] if $F'''(R_0) \neq 0$.

In summary, although removing the singularities is definitely an option, and a non-singular evolution can be realized, since singular evolution is a possibility too in this work we are interested to investigate the resulting cosmological behavior. Moreover, we desire to show that in some cases a singular evolution can be consistent and in agreement with the standard observed Universe history.

3 Two-field scalar-tensor gravity: specific examples

The use of two scalar fields offers a great number of possibilities for cosmological evolution, with various diverse features that were absent in the case of a single scalar field. In this

section we are interested in reconstructing specific two-field models that may still exhibit some singularity types, in particular Type II and Type IV.

3.1 Generic examples of two-field scalar-tensor theories exhibiting Type II or Type IV singularities

Let us start by considering the Hubble rate to have the very general form

$$H(t) = f_1(t) + f_2(t) (t_s - t)^\alpha. \quad (3.1)$$

We can choose the arbitrary functions $f_1(t)$ and $f_2(t)$ to be smooth and differentiable functions of t . In the following, and without loss of generality, we shall consider the case where α is given by two integers n and m , namely

$$\alpha = \frac{n}{2m+1}, \quad (3.2)$$

and specify furthermore the value of the integer n when this is needed. For more general values of the parameter α we may extend (3.1) to

$$H(t) = f_1(t) + f_2(t) |t_s - t|^\alpha. \quad (3.3)$$

Concerning the kinetic functions $\omega(\phi)$ and $\eta(\phi)$ of the two fields that appear in action (2.3), we use

$$\omega(\phi) = -\frac{2}{\kappa^2} f'_1(\phi), \quad \eta(\chi) = -\frac{2}{\kappa^2} \left[f'_2(\chi) (t_s - \chi)^\alpha + \alpha f_2(\chi) (t_s - \chi)^{\alpha-1} \right], \quad (3.4)$$

which implies that the function $\alpha_1(x)$ appearing in (2.37) is of the form

$$\alpha_1(x) = \sqrt{\left[f_2(x) (t_s - x)^\alpha + \alpha f_2(x) (t_s - x)^{\alpha-1} \right]^2 - [H'(x)]^2}. \quad (3.5)$$

Here, the variable x can be either the scalar field ϕ or χ . Having chosen $\alpha_1(x)$ as in expression (3.5) allows to specify the final form of the function $\tilde{f}(\phi, \chi)$, defined in (2.38), and indeed in the case at hand becomes

$$\tilde{f}(\phi, \chi) = f_1(\phi) + f_2(\chi) (t_s - \chi)^\alpha. \quad (3.6)$$

The corresponding scalar potential of (2.40) reads

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left\{ 3 [f_1(\phi) + f_2(\chi) (t_s - \chi)^\alpha]^2 + f'_1(\phi) + f'_2(\chi) (t_s - \chi)^\alpha + \alpha f_2(\chi) (t_s - \chi)^{\alpha-1} \right\}. \quad (3.7)$$

Additionally, the kinetic functions from (3.4) now become

$$\begin{aligned} \omega(\phi) &= -\frac{2f'_1(t)}{\kappa^2}, \\ \eta(\chi) &= -\frac{2 \left[\alpha (t_s - \chi)^{-1+\alpha} f_2(\chi) + (t_s - \chi)^\alpha f'_2(t) \right]}{\kappa^2}. \end{aligned} \quad (3.8)$$

Finally, from the form of the Hubble rate (3.3), we can straightforwardly extract the total equation-of-state parameter of the universe (2.23) as

$$w_{\text{eff}} = -1 - \frac{2 [f_1'(t) + (-t + t_s)^\alpha f_2'(t) - (-t + t_s)^{-1+\alpha} \alpha f_2(t)]}{3 [f_1(t) + (-t + t_s)^\alpha f_2(t)]^2}. \quad (3.9)$$

As we already mentioned, the functions $f_1(t)$ and $f_2(t)$ are arbitrary functions, so a convenient choice might lead to interesting results, regarding the cosmological evolution. Assuming that the Hubble rate is equal to $H(t) = f_1(t) + f_2(t)$, one convenient choice for $f_1(t)$ and $f_2(t)$ could be

$$f_1(t) = f^\phi(\phi), \quad f_2(t) = f^\chi(\chi), \quad (3.10)$$

where

$$f^\phi(\phi) = f_1^\phi(\phi) + f_2^\phi(\phi) (t_1 - \phi)^\alpha, \quad f^\chi(\chi) = f_1^\chi(\chi) + f_2^\chi(\chi) (t_2 - \chi)^\beta, \quad (3.11)$$

and where $f_1^\phi(\phi)$, $f_2^\phi(\phi)$, $f_1^\chi(\chi)$, and $f_2^\chi(\chi)$ are smooth functions of the scalar fields. Then, in terms of the functions appearing in (3.11), the Hubble rate is written as

$$H(t) = f^\phi(t) + f^\chi(t) = f_1^\phi(t) + f_1^\chi(t) + f_2^\phi(t) (t_1 - \phi)^\alpha + f_2^\chi(t) (t_2 - \phi)^\beta. \quad (3.12)$$

An interesting cosmological scenario can be realized if we choose $t_1 \ll t_2$, $\alpha > 1$ and $\beta < 0$. In this case, t_1 could correspond to a time around the inflation era, while t_2 to the dark energy epoch, present or future. In addition, when $\alpha > 1$ and $\beta < 0$ it easily follows from the Hubble rate (3.12) that at t_1 the physical system of the two scalars experiences a Type IV singularity, while at t_2 , it experiences a Type I (Big Rip) singularity, if $\beta < -1$, or a Type III, if $-1 < \beta < 0$. It is worthy to further specify the functions appearing in (3.11) and quantitatively examine the cosmological evolution and the resulting physical picture. As an example we choose

$$f^\phi(\phi) = \frac{f_1}{\sqrt{t_0^2 + \phi^2}} + \frac{f_2 \phi^2}{t_0^4 + \phi^4} (t_1 - \phi)^\alpha, \quad f^\chi(\chi) = f_3 (t_2 - \chi)^\beta, \quad (3.13)$$

where t_0 , f_1 , f_2 , and f_3 are considered to be positive constants, so that $H(t) > 0$. For the choice (3.13), the Hubble rate becomes

$$H(t) = \frac{f_1}{\sqrt{t^2 + t_0^2}} + \frac{f_2 t^2 (-t + t_1)^\alpha}{t^4 + t_0^4} + f_3 (-t + t_2)^\beta, \quad (3.14)$$

and the corresponding kinetic functions of the two scalars read

$$\begin{aligned} \omega(\phi) &= -\frac{2}{\kappa^2} \left[-\frac{f_1 t}{(t^2 + t_0^2)^{3/2}} - \frac{4 f_2 t^5 (-t + t_1)^\alpha}{(t^4 + t_0^4)^2} + \frac{2 f_2 t (-t + t_1)^\alpha}{t^4 + t_0^4} - \frac{f_2 t^2 (-t + t_1)^{-1+\alpha} \alpha}{t^4 + t_0^4} \right], \\ \eta(\chi) &= \frac{2 f_3 (-t + t_2)^{-1+\beta} \beta}{\kappa^2}. \end{aligned} \quad (3.15)$$

Since the parameter β is assumed to be $\beta < 0$, the kinetic function $\eta(\chi)$ is negative, and therefore χ is a ghost field. On the contrary, $\omega(\phi)$ is positive and therefore ϕ is a canonical scalar field, as long as $f_2 \ll f_1$. The corresponding $\tilde{f}(\phi, \chi)$ function from (3.6) writes as

$$\tilde{f}(\phi, \chi) = \frac{f_1}{\sqrt{t_0^2 + \phi^2}} + \frac{f_2(t_1 - \phi)^\alpha \phi^2}{t_0^4 + \phi^4} + f_3(t_2 - \chi)^\beta, \quad (3.16)$$

and therefore the scalar potential (3.7) is given by

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left[-\frac{f_1 \phi}{(t_0^2 + \phi^2)^{3/2}} - \frac{4f_2(t_1 - \phi)^\alpha \phi^5}{(t_0^4 + \phi^4)^2} + \frac{2f_2(t_1 - \phi)^\alpha \phi}{t_0^4 + \phi^4} - \frac{f_2 \alpha (t_1 - \phi)^{-1+\alpha} \phi^2}{t_0^4 + \phi^4} \right] \\ + \frac{3}{\kappa^2} \left\{ \left[\frac{f_1}{\sqrt{t_0^2 + \phi^2}} + \frac{f_2(t_1 - \phi)^\alpha \phi^2}{t_0^4 + \phi^4} + f_3(t_2 - \chi)^\beta \right]^2 - f_3 \beta (t_2 - \chi)^{-1+\beta} \right\}. \quad (3.17)$$

Finally, the total equation-of-state parameter corresponding to the Hubble rate (3.14) reads

$$w_{\text{eff}} = -1 - \frac{2 \left[-\frac{f_1 t}{(t^2 + t_0^2)^{3/2}} - \frac{4f_2 t^5 (-t + t_1)^\alpha}{(t^4 + t_0^4)^2} + \frac{2f_2 t (-t + t_1)^\alpha}{t^4 + t_0^4} - \frac{f_2 t^2 (-t + t_1)^{-1+\alpha} \alpha}{t^4 + t_0^4} - f_3 (-t + t_2)^{-1+\beta} \beta \right]}{3 \left[\frac{f_1}{\sqrt{t^2 + t_0^2}} + \frac{f_2 t^2 (-t + t_1)^\alpha}{t^4 + t_0^4} + f_3 (-t + t_2)^\beta \right]^2}. \quad (3.18)$$

In the early universe $f^\phi(t)$ dominates in $H(t)$ in (3.12), and therefore the inflationary era occurs, with a Type IV singularity at $t = t_1$. In the late-time universe $f^\chi(t)$ dominates in $H(t)$ in (3.12), and the universe will end with a Type I (Big Rip) singularity ($\beta < -1$) or Type III singularity ($-1 < \beta < 0$) at $t = t_2$. Having made these assumptions, it is worthy to study the behavior of the EoS (3.18) as a function of the cosmic time. Let us first consider time scales near the Type IV singularity, thus $t \sim t_1$. In such a case, by disregarding subdominant terms, the EoS takes the form

$$w_{\text{eff}} \simeq -1 - \frac{2 \left[-\frac{f_1 t}{(t_0^2)^{3/2}} - f_3 (t_2)^{-1+\beta} \beta \right]}{3 \left[\frac{f_1}{\sqrt{t_0^2}} + f_3 (t_2)^\beta \right]^2}, \quad (3.19)$$

and since $t_2 \gg 1$ and $\beta < 0$ it finally becomes

$$w_{\text{eff}} \simeq -1 + \frac{2t}{3f_1 t_0}, \quad (3.20)$$

which describes quintessential inflation. In the case where $t_0 \gg 1$, the EoS (3.20) becomes approximately $w_{\text{eff}} \simeq -1$, i.e. it is effectively a cosmological constant. Furthermore, with regards to the late-time behavior, which corresponds to cosmic times near t_2 , we shall assume that $t_2 > t_0$ and also that α is of the form given in (3.2), namely $\alpha = n/(2m + 1)$. For n even, and $2 < \alpha < 3$, the effective equation of state reads

$$w_{\text{eff}} \simeq -1 + \frac{2f_3 (-t + t_2)^{-1+\beta} \beta}{3 [f_2 t^{-2+\alpha} + f_3 (-t + t_2)^\beta]^2}, \quad (3.21)$$

which is $w_{\text{eff}} < -1$ when $t < t_2$. Therefore, eventually this case describes phantom dark energy at a time before the singularity. Additionally, the case $\alpha > 3$ and n even is slightly more complicated. In this case we set $x = -t + t_2$, and the EoS becomes

$$w_{\text{eff}} \simeq -1 - \frac{2(t_2 - x)^{1-\alpha}(-2 + \alpha)}{3f_2}, \quad (3.22)$$

which for $t < t_2$ (or $x > 0$) is $w_{\text{eff}} < -1$, which again describes phantom evolution. In the case where n is an even integer and $\alpha < 3$, the EoS reads

$$w_{\text{eff}} \simeq -1 + \frac{2f_3(-t + t_2)^{-1+\beta}\beta}{3[f_2t^{-2+\alpha} - f_3(-t + t_2)^\beta]^2}, \quad (3.23)$$

which describes phantom acceleration (recall that $\beta < 0$), while for $\alpha > 3$ we have

$$w_{\text{eff}} \simeq -1 - \frac{2(t_2 - x)^{1-\alpha}(2 + \alpha)}{3f_2}, \quad (3.24)$$

which describes phantom acceleration too. In summary, with two scalar fields we may realize a plethora of cosmological expansions, with finite-time singularities present in the cosmological evolution.

We close this subsection with another model with an interesting cosmological evolution. In particular, we consider the Hubble rate to be

$$H(t) = f_1^\phi(t) + f_1^\chi(t) + f_2(t)(t_s - t)^\alpha. \quad (3.25)$$

The function $f_1^\phi(t)$ can be chosen in such a way that it describes the (smooth) inflation era, and in addition the function $f_1^\chi(t)$ can be chosen to describe the present time accelerating expansion of the Universe. We can consider the following two classes of models, in which the functions are:

$$(A): f^\phi(\phi) = f_1^\phi(\phi) + f_2(\phi)(t_s - \phi)^\alpha, \quad f^\chi(\chi) = f_1^\chi(\chi), \quad (3.26)$$

$$(B): f^\phi(\phi) = f_1^\phi(\phi), \quad f^\chi(\chi) = f_1^\chi(\chi) + f_2(\chi)(t_s - \chi)^\alpha. \quad (3.27)$$

Obviously, both models (A) and (B) generate identical evolution of the Universe's expansion, since the Hubble rate is the same. In model (A), the scalar field ϕ can be selected in order to generate both the inflationary era and the finite time singularity. In addition, in model (B) χ can generate both the present accelerating expansion, as well as the finite time singularity. We may choose t_s to be large enough, in order to correspond to the far future, and moreover the function $f_2(t)$ can be chosen in order to give a small contribution to the present cosmological evolution of the Universe, and hence it is difficult for its contribution to be observed. In the following, we shall present a model for which this scenario can be realized.

For brevity, let us consider model (3.26). The Hubble rate is simplified as

$$H(t) = f_1^\phi(t) + (-t + t_1)^\alpha f_2^\phi(t) + f_1^\chi(t), \quad (3.28)$$

and the kinetic functions of the scalar-tensor theory read

$$\omega(\phi) = -\frac{2 \left[-(-t + t_1)^{-1+\alpha} \alpha f_2^\phi(t) + f_1^\phi(t) + (-t + t_1)^\alpha f_2^\phi(t) \right]}{\kappa^2}, \quad (3.29)$$

while

$$\eta(\chi) = -\frac{2f_1^\chi(t)}{\kappa^2}. \quad (3.30)$$

The corresponding scalar potential (3.7) becomes

$$V(\phi, \chi) = -\frac{\alpha}{\kappa^2} \left\{ (t_1 - \phi)^{-1+\alpha} f_2^\phi(\phi) + 3 \left[f_1^\phi(\phi) + (t_1 - \phi)^\alpha f_2^\phi(\phi) + f_1^\chi(\chi) \right]^2 + f_1^\phi(\phi) + (t_1 - \phi)^\alpha f_2^\phi(\phi) + f_1^\chi(\chi) \right\}, \quad (3.31)$$

while the total EoS writes as

$$w_{\text{eff}} = -1 - \frac{2 \left[-(-t + t_1)^{-1+\alpha} \alpha f_2^\phi(t) + f_1^\phi(t) + (-t + t_1)^\alpha f_2^\phi(t) + f_1^\chi(t) \right]}{3 \left[f_1^\phi(t) + (-t + t_1)^\alpha f_2^\phi(t) + f_1^\chi(t) \right]^2}. \quad (3.32)$$

A convenient choice for the functions appearing in the Hubble rate (3.28) is

$$f_1^\phi(t) = \frac{f_1}{\sqrt{t^2 + t_0^2}}, \quad f_2^\phi(t) = e^{-t\beta}, \quad f_1^\chi(t) = \frac{f_2 t^2}{t^4 + t_0^4}, \quad (3.33)$$

and in this case we finally acquire

$$w_{\text{eff}} = -1 - \frac{2 \left[-\frac{f_1 t}{(t^2 + t_0^2)^{3/2}} - \frac{4f_2 t^5}{(t^4 + t_0^4)^2} + \frac{2f_2 t}{t^4 + t_0^4} - e^{-t\beta} (-t + t_1)^{-1+\alpha} \alpha - e^{-t\beta} (-t + t_1)^\alpha \beta \right]}{3 \left[\frac{f_1}{\sqrt{t^2 + t_0^2}} + \frac{f_2 t^2}{t^4 + t_0^4} + e^{-t\beta} (-t + t_1)^\alpha \right]^2}. \quad (3.34)$$

Thus, we can now study its behavior, focusing mainly at times corresponding to the inflationary era, as well as at late times.

We first consider that t_1 corresponds to the inflationary regime, and that at this point a Type IV singularity occurs. This implies that $\alpha > 1$ and moreover we assume that $\beta > 0$. Furthermore, if $t_0 > t_1$ then the EoS at early times reads

$$w_{\text{eff}} = -1 + \frac{2t(-2f_2 + f_1 t_0)}{3f_1^2 t_0^2}. \quad (3.35)$$

In the case where $f_1 t_0 > 2f_2$, the Universe accelerates in a quintessential way near the Type IV singularity at t_1 . In addition, if $t_0 \gg 1$ then the second term in (3.35) is negligible, and therefore $w_{\text{eff}} \simeq -1$, which corresponds to a de Sitter expansion. Definitely, if $f_1 t_0 < 2f_2$ then the EoS (3.35) describes phantom inflation.

Let us now focus at late times, where the effective EoS behaves as

$$w_{\text{eff}} \simeq -1 + \frac{2t}{3f_1 \sqrt{t^2 + t_0^2}}, \quad (3.36)$$

which describes quintessential acceleration. In the case where $t \simeq t_0$, and $t_0 \gg 1$, relation (3.36) becomes a nearly de Sitter late-time acceleration, with $w_{\text{eff}} \simeq -1$. It is worthy to say that the scalar field ϕ is the one with the positive kinetic term in all cases, with $\omega(\phi)$ being equal to

$$\omega(\phi) = -\frac{2 \left[-\frac{f_1 t}{(t^2 + t_0^2)^{3/2}} - e^{-t\beta} (-t + t_1)^{-1+\alpha} \alpha - e^{-t\beta} (-t + t_1)^\alpha \beta \right]}{\kappa^2}, \quad (3.37)$$

while $\eta(\chi)$ reads

$$\eta(\chi) = \frac{4f_2 (t^5 - t t_0^4)}{(t^4 + t_0^4)^2 \kappa^2}, \quad (3.38)$$

and since $t_0 \gg 1$ it can be negative for a wide range of t values (we assume that $f_2 > 0$). Therefore, the use of multiple scalar fields can significantly increase the cosmological scenarios for the physical system described by a specific Hubble rate.

3.2 Slow-roll with two-field scalar-tensor theories

In this subsection we qualitatively describe how the slow-roll approximation is affected by the use of two scalar fields describing the cosmological evolution. In addition, we discuss how the singularities affect the general slow-roll evolution and also we investigate when these can lead to inconsistencies. In principle, the results can be model-dependent, but using some illustrative examples we shall see how a singularity can affect the inflation parameters. Our study will remain at a qualitative level, since a detailed investigation would require the analysis of two- and three- point statistics of field perturbations, which can be related to the two- and three- point statistics of the curvature perturbations on uniform density hypersurfaces. This could be achieved by using the δN -formalism [119–127], however this detailed study lies beyond the illustrative scopes of this article. For details the reader is referred to [119–127].

The most important inconsistency that the singularities might cause, is traced in the definition of the e -folding number N . The inconsistency occurs when the singularity appears during the inflationary era, even in the case the singularity is of Type IV. Therefore, we shall assume that the singularity occurs at exactly the time when inflation ends. In the end of this subsection we shall discuss in detail this problematic issue.

In general, single-field and multi-field inflation are very different, when the corresponding scenarios are viewed as dynamical systems. In the single-field case, the phase space formed by the slow-roll solution is one-dimensional, implying that the slow-roll solution is the unique attractor of the dynamical system. On the other hand, in the case where two-field inflation is studied, the phase space is two dimensional and the stable trajectories form an infinite countable space [119–127]. Hence, the field values at the end of inflation will generally depend on the choice of the classical field trajectory. The corresponding e -folding number will determine the evolution along one of these classical trajectories [119–127]. In order to see this in a qualitative way, in this subsection we shall investigate when the slow-roll condition is violated in the cases of single-field and two-field inflation. The violation of

the slow-roll condition will actually determine when inflation ends. We quantify our results by studying the behavior of the slow-roll inflation parameter ϵ .

As a toy single-field example we shall consider a viable inflationary model, which belongs to the class of large-field inflationary realizations. In a previous article we performed a thorough analysis on this type of scenarios [20], and thus we now focus on the qualitative analysis of the slow-roll parameter ϵ . The model consists of a canonical scalar field φ , and the potential for large field values is given by

$$V(\varphi) \simeq d(\varphi - t_1)^n, \quad (3.39)$$

with t_1 the time at which the Type IV singularity occurs, which for consistency is chosen to coincide with the time that inflation ends. As was demonstrated in [20] the Type IV singularity can be consistently accommodated in the cosmological evolution of the model under study. The slow-roll parameter ϵ in the case of a single canonical scalar field reads as

$$\epsilon = \frac{1}{2\kappa^2} \left[\frac{V'(\varphi)}{V(\varphi)} \right], \quad (3.40)$$

and thus for the potential (3.39) it becomes

$$\epsilon(\varphi) = \frac{n^2}{2\kappa^2(t_s - \varphi)^2}. \quad (3.41)$$

It is easy to analytically calculate when inflation ends, which happens at a field value φ_e for which $\epsilon(\varphi_e) \sim 1$. Indeed, the slow-roll condition is violated when the scalar field exceeds the critical value φ_e , namely

$$\varphi_e - t_1 = \frac{n}{\sqrt{2}\kappa}. \quad (3.42)$$

In order to acquire an overall picture of the phenomenon, in Fig. 1 we depict the behavior of $\epsilon(\varphi)$ as a function of φ . Notice that the large field values correspond to early times, and as the field value decreases, the cosmological time increases. Hence, the x -axis should be viewed backwards, as time increases. As we can see, the slow-roll condition is violated at the field value $\varphi_e = t_1 + \frac{n}{\sqrt{2}\kappa}$, which is indicated by a dot on the graph. Thus, in the single-scalar case the ending of slow roll is characterized by a single point in the corresponding phase space. This fact has a direct impact on the slow-roll parameter $\epsilon(\varphi)$, as can be seen in Fig. 1.

Let us now come to the two-field case, which as we show is more involved. In order to simplify things we choose a simple example, and we consider the Hubble rate to be

$$H(t) = f_1(t) + f_2(t), \quad (3.43)$$

with $f_1(t)$ and $f_2(t)$ being equal to

$$f_1(t) = \nu_1 (-t + t_1)^\alpha, \quad f_2(t) = \nu_2 (-t + t_1)^\alpha, \quad (3.44)$$

where ν_1 and ν_2 are parameters. In this case the Type IV singularity occurs if $\alpha > 1$. Thus, we can accommodate the Type IV singularity in a generalized two-field scalar-tensor

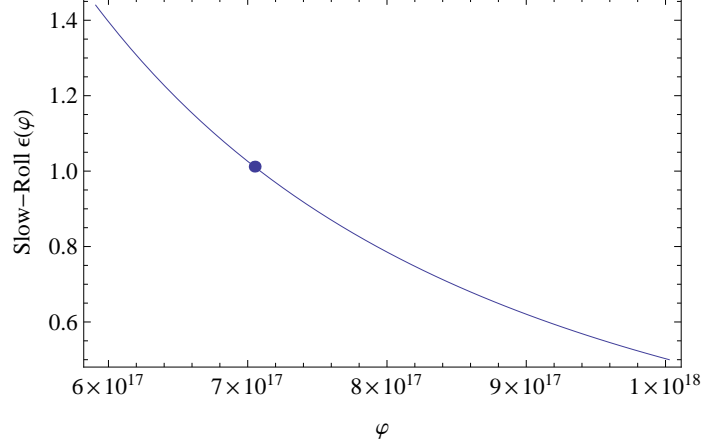


Figure 1. The slow-roll parameter $\epsilon(\varphi)$ as a function of the scalar field φ , for the potential (3.39) for $n \sim 1.95$. As the scalar field decreases the cosmological time is assumed to increase. The dot corresponds to the critical field value $\varphi_e = t_1 + \frac{n}{\sqrt{2\kappa}}$, at which the slow-roll condition is violated.

theory with action given in (2.32), by following the reconstruction method of the previous section, with $\alpha_1(x)$ chosen as in (3.5). In particular, the kinetic functions $\omega(\phi)$ and $\eta(\chi)$ are equal to

$$\omega(\phi) = -\frac{2}{\kappa^2} f'_1(\phi), \quad \eta(\chi) = -\frac{2}{\kappa^2} f'_2(\chi), \quad (3.45)$$

and by substituting the functions $f_{1,2}(t)$ from (3.44) we acquire

$$\omega(\phi) = \frac{2\nu_1(-t+t_1)^{\alpha-1}\alpha}{\kappa^2}, \quad \eta(\chi) = \frac{2\nu_2(-t+t_1)^{\alpha-1}\alpha}{\kappa^2}. \quad (3.46)$$

Therefore, the corresponding scalar potential $V(\phi, \chi)$ writes as

$$\begin{aligned} V(\phi, \chi) = & -\frac{\nu_1\alpha(t_1-\phi)^{-1+\alpha}}{\kappa^2} + \frac{3\nu_1^2(t_1-\phi)^{2\alpha}}{\kappa^2} - \frac{\nu_2\alpha(t_1-\chi)^{-1+\alpha}}{\kappa^2} \\ & + \frac{6\nu_1\nu_2(t_1-\phi)^\alpha(t_1-\chi)^\alpha}{\kappa^2} + \frac{3\nu_2^2(t_1-\chi)^{2\alpha}}{\kappa^2}. \end{aligned} \quad (3.47)$$

We assume that t_1 is identical with the cosmological time when inflation ends, and thus it is a very small number. In addition, we assume that inflation occurs for large field values, and thus at the time when inflation occurs, $t_1 - \phi \gg 1$ and $t_1 - \chi \gg 1$. Since $a > 1$, the potential can be simplified to a great extent by keeping the dominant terms, becoming

$$V(\phi, \chi) = \frac{3\nu_1^2(t_1-\phi)^{2\alpha}}{\kappa^2} + \frac{6\nu_1\nu_2(t_1-\phi)^\alpha(t_1-\chi)^\alpha}{\kappa^2} + \frac{3\nu_2^2(t_1-\chi)^{2\alpha}}{\kappa^2}. \quad (3.48)$$

As a next step, we can transform the above two-field scalar-tensor theory to one that contains only canonical scalar fields. Indeed, by making the transformation

$$t_1 - \sigma = \int^\chi \sqrt{\eta(\chi)} d\chi, \quad t_1 - \varphi = \int^\phi \sqrt{\omega(\phi)} d\phi, \quad (3.49)$$

we can extract the identifications

$$-\phi + t_1 = \alpha_1 (\varphi - t_1)^{\frac{2}{\alpha+1}}, \quad -\chi + t_1 = \alpha_2 (\sigma - t_1)^{\frac{2}{\alpha+1}}, \quad (3.50)$$

where α_1 and α_2 stand for

$$\alpha_1 = \left[\frac{2\sqrt{2\alpha\nu_1}}{(1+\alpha)\kappa} \right]^{\frac{2}{\alpha+1}}, \quad \alpha_2 = \left[\frac{2\sqrt{2\alpha\nu_2}}{(1+\alpha)\kappa} \right]^{\frac{2}{\alpha+1}}. \quad (3.51)$$

Hence, in terms of the canonical scalar fields φ and σ , the simplified scalar potential (3.48) reads

$$V(\varphi, \sigma) = \frac{3\nu_1^2(\alpha_1)^{2\alpha}(\varphi - t_1)^{\frac{4\alpha}{\alpha+1}}}{\kappa^2} + \frac{6\nu_1\nu_2(\alpha_1\alpha_2)^\alpha(\varphi - t_1)^{\frac{2\alpha}{\alpha+1}}(\sigma - t_1)^{\frac{2\alpha}{\alpha+1}}}{\kappa^2} + \frac{3\nu_2^2(\alpha_2)^{2\alpha}(\sigma - t_1)^{\frac{4\alpha}{\alpha+1}}}{\kappa^2}, \quad (3.52)$$

while the canonical two-field action is simply

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - V(\varphi, \sigma) \right\}. \quad (3.53)$$

Having the scalar potential (3.52) at hand, we can qualitatively investigate when the slow-roll approximation is violated. Practically this occurs when the slow-roll parameter ϵ becomes $\epsilon \gtrsim 1$. In the case of a scalar-tensor theory with two canonical scalar fields, the slow-roll parameter ϵ is equal to [119–127]:

$$\epsilon(\varphi, \sigma) = \epsilon_\varphi + \epsilon_\sigma, \quad (3.54)$$

with ϵ_φ and ϵ_σ reading as

$$\epsilon_\varphi = \frac{1}{2\kappa^2} \left[\frac{\partial_\varphi V(\varphi, \sigma)}{V(\varphi, \sigma)} \right]^2, \quad \epsilon_\sigma = \frac{1}{2\kappa^2} \left[\frac{\partial_\sigma V(\varphi, \sigma)}{V(\varphi, \sigma)} \right]^2. \quad (3.55)$$

Thus, by using the potential (3.52) we may directly evaluate the slow-roll parameter ϵ and study its evolution as a function of the two scalar fields. Assuming again large field values for small cosmological times, which decrease as the cosmic time approaches the end of inflation, in Fig. 2 we present the behavior of the slow-roll parameter ϵ as a function of the scalar fields. As we observe the slow-roll approximation is violated for a combination of values of the two fields, as expected.

Before closing this subsection, we shall investigate the impact of the singularities on the cosmological parameters of the two-field model. In particular, we are interested in the cases where the singularities may lead to inconsistencies. As we mentioned earlier, the main inconsistency is traced in the definition of the e -fold number N , which is defined as

$$N = - \int_{t_f}^t H(t_*) dt, \quad (3.56)$$

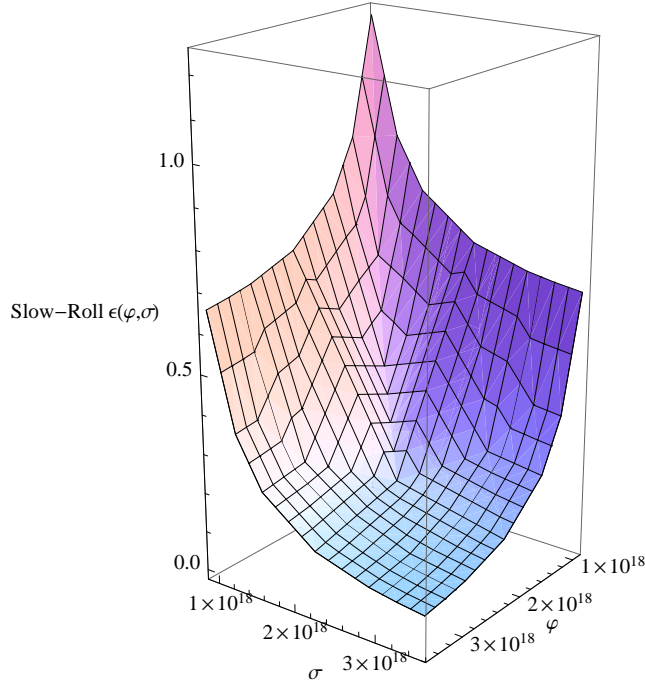


Figure 2. The slow-roll parameter $\epsilon(\varphi, \sigma)$ as a function of the scalar fields φ and σ , for the choice (3.44) for $\nu_1 = \nu_2$ and $\alpha \sim 1.1$. As the scalar field values decrease, the cosmological time is assumed to increase.

where t_* is the Hubble exit during inflation. The final time t_f corresponds to a uniform-density hypersurface, at which inflation is considered to end. By taking into account (3.43) and (3.44), the e -folding number writes as

$$N = \frac{(\nu_1 + \nu_2) [(t_1 - t_f)^{\alpha+1} - (-t_* + t_1)^{\alpha+1}]}{\alpha + 1}. \quad (3.57)$$

Hence, it is obvious that when $\alpha < -1$ the above expression is always singular when $t_f = t_1$, due to the last term. However, if the singularity is of Type IV there is no inconsistency, which is the main point of this subsection.

Finally, as we mentioned earlier too, we note that the complete analysis of the e -folding number for the two-field case can be quite complicated, and a detailed investigation would require the use of the δN approximation. However, such a study stretches beyond the scope of the present work, and the reader is referred to [119–127] and references therein.

3.3 Singular inflation with singular dark energy

In this subsection we are interested in constructing a cosmological scenario where singular inflation gives rise to radiation and matter eras, and eventually to a dark energy epoch that results in a Big-Rip singularity. In order to succeed this, and following the previous subsections, we use two scalar fields, and thus this is another indication of the capabilities of such a model.

Having in mind the discussion on the singularity types of the Hubble rate (2.14), we deduce that a Hubble rate that captures the above features, namely a singular inflation with a type IV singularity and a dark energy epoch resulting to a Big Rip, writes as

$$H(t) = c_0 + b_0 (t_s - t)^\alpha + d_0 (t_B - t)^\gamma , \quad (3.58)$$

where $\alpha > 1$ and $\gamma < -1$, with c_0, b_0, d_0 constants, and where t_s and t_B are respectively the time of the inflationary type IV singularity and of the Big Rip. We need to note that α is assumed to take values of (3.2), therefore complex values for the Hubble rate are avoided for the physically interesting interval $t_s < t < t_B$. Practically, since the denominator of the fraction appearing in (3.2) is an odd integer always, then this procedure is always valid. Definitely, there are extra complex conjugate branches too, but we only focus on the real negative one.

Assuming as usual that the two scalar fields ϕ and χ depend only on the cosmic-time coordinate t , and also that the spacetime is described by a flat FRW metric of the form (2.1), we extract the Friedmann equations in the form

$$\omega(\phi)\dot{\phi}^2 + \eta(\chi)\dot{\chi}^2 = -\frac{2}{\kappa^2}\dot{H} , \quad (3.59)$$

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left(3H^2 + \dot{H} \right) . \quad (3.60)$$

Following the procedure of the previous subsections we impose the parametrization

$$\begin{aligned} \omega(t) + \eta(t) &= -\frac{2}{\kappa^2}f'(t) , \\ V(t) &= \frac{1}{\kappa^2} [3f(t)^2 + f'(t)] , \end{aligned} \quad (3.61)$$

which is consistent with the explicit solution of (3.59) and (3.60):

$$\phi = \chi = t, \quad H = f(t) . \quad (3.62)$$

A convenient choice for the kinetic functions would be

$$\begin{aligned} \omega(\phi) &= -\frac{2}{\kappa^2} \left\{ f'(\phi) - \sqrt{\alpha_1(\phi)^2 + f'(\phi)^2} \right\} > 0 , \\ \eta(\chi) &= -\frac{2}{\kappa^2} \sqrt{\alpha_1(\chi)^2 + f'(\chi)^2} < 0 , \end{aligned} \quad (3.63)$$

with $\alpha_1(x)$ an arbitrary function of its argument. It proves useful to define a new function $\tilde{f}(\phi, \chi)$ through

$$\tilde{f}(\phi, \chi) \equiv -\frac{\kappa^2}{2} \left[\int d\phi \omega(\phi) + \int d\chi \eta(\chi) \right] , \quad (3.64)$$

which has the characteristic property $\tilde{f}(t) = f(t)$ which fixes the constants of integration in (3.64). Assuming that $V(\phi, \chi)$ is given in terms of the function $\tilde{f}(\phi, \chi)$ as

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left[3\tilde{f}(\phi, \chi)^2 + \frac{\partial \tilde{f}(\phi, \chi)}{\partial \phi} + \frac{\partial \tilde{f}(\phi, \chi)}{\partial \chi} \right] , \quad (3.65)$$

we find that in addition to the Friedmann equations given in (3.59) and (3.60) we have

$$0 = \omega(\phi)\ddot{\phi} + \frac{1}{2}\omega'(\phi)\dot{\phi}^2 + 3H\omega(\phi)\dot{\phi} + \frac{\partial\tilde{V}(\phi,\chi)}{\partial\phi}, \quad (3.66)$$

$$0 = \eta(\chi)\ddot{\chi} + \frac{1}{2}\eta'(\chi)\dot{\chi}^2 + 3H\eta(\chi)\dot{\chi} + \frac{\partial\tilde{V}(\phi,\chi)}{\partial\chi}. \quad (3.67)$$

Hence, using the above kinetic functions $\omega(\phi)$, $\eta(\chi)$ and the scalar potential $V(\phi, \chi)$ we obtain a two-field scalar-tensor model with cosmological evolution of the form (3.58).

As a next step we need to impose an ansatz for the function $\alpha_1(x)$ appeared in (3.63). We choose it as

$$\alpha_1(x) = \sqrt{b_0\alpha(-x + t_s)^{\alpha-1} - H'(x)^2}, \quad (3.68)$$

and therefore (3.63) give

$$\begin{aligned} \omega(\phi) &= \frac{2 \left[b_0\alpha(t_s - \phi)^{\alpha-1} + d_0\gamma(t_B - \phi)^{\gamma-1} + \sqrt{b_0\alpha(t_s - \phi)^{\alpha-1}} \right]}{\kappa^2}, \\ \eta(\chi) &= -\frac{2\sqrt{b_0\alpha(t_s - \chi)^{-1+\alpha}}}{\kappa^2}. \end{aligned} \quad (3.69)$$

As expected, one of the fields is canonical while the other one is ghost. The corresponding function $\tilde{f}(\phi, \chi)$ from (3.64) becomes

$$\tilde{f}(\phi, \chi) = -b_0(t_s - \phi)^\alpha - d_0(t_B - \phi)^\gamma - \frac{2\sqrt{b_0\alpha}(t_s - \phi)^{\frac{\alpha+1}{2}}}{(\alpha+1)} - \frac{2\sqrt{b_0\alpha}(t_s - \chi)^{\frac{\alpha+1}{2}}}{(\alpha+1)}, \quad (3.70)$$

and thus the scalar potential $V(\phi, \chi)$ from (3.65) becomes

$$\begin{aligned} V(\phi, \chi) &= \frac{b_0\alpha(t_s - \phi)^{\alpha-1} + d_0\gamma(t_B - \phi)^{\gamma-1}}{\kappa^2} \\ &+ \frac{2\sqrt{b_0\alpha}(t_s - \phi)^{\frac{\alpha-1}{2}} + \sqrt{b_0\alpha}(\alpha-1)(t_s - \phi)^{\frac{\alpha-1}{2}}}{(1+\alpha)\kappa^2} \\ &+ \left[\frac{2\sqrt{b_0\alpha}(t_s - \chi)^{\frac{\alpha-1}{2}}}{(1+\alpha)\kappa^2} - \frac{\sqrt{b_0\alpha}(\alpha-1)(t_s - \chi)^{\alpha-1}}{(1+\alpha)\kappa^2} \right] \\ &+ \frac{3}{\kappa^2} \left\{ b_0(t_s - \phi)^\alpha + d_0(t_B - \phi)^\gamma + \frac{2\sqrt{b_0\alpha}(t_s - \phi)^{\frac{\alpha+1}{2}}}{(\alpha+1)} + \frac{2\sqrt{b_0\alpha}(t_s - \chi)^{\frac{\alpha+1}{2}}}{(\alpha+1)} \right\}^2. \end{aligned} \quad (3.71)$$

In summary, the above two-field model can give rise to the Hubble rate (3.58), which corresponds to a universe with a singular (of type IV) inflation, resulting in a dark-energy phase that ends with a Big Rip.

3.4 Barrow's two-field model

In this subsection, for completeness, we present the realization of the singular inflation in the Barrow's two-field model [74]. This model can exhibit the Type IV singularity

consistently incorporated in its scalar-tensor theoretical framework. In particular, we consider a Hubble rate of the form

$$H(t) = b_0 (t_s - t)^\alpha. \quad (3.72)$$

We choose the arbitrary function $\alpha_1(x)$ appearing in (2.37) to be

$$\alpha_1(x) = \sqrt{2\alpha b_0}(-x + t_s)^{(\alpha-1)/2}, \quad (3.73)$$

and substituting this into (2.37) we can easily obtain the corresponding kinetic functions $\omega(\phi)$ and $\eta(\chi)$ as

$$\begin{aligned} \omega(\phi) &= -\frac{2 \left[-b_0 \alpha (t_s - \phi)^{-1+\alpha} - \sqrt{b_0 \alpha} (t_s - \phi)^{\frac{-1+\alpha}{2}} \right]}{\kappa^2}, \\ \eta(\chi) &= -\frac{2\sqrt{b_0 \alpha} (t_s - \phi)^{\frac{-1+\alpha}{2}}}{\kappa^2}. \end{aligned} \quad (3.74)$$

In the case at hand, and assuming that α has the form (2.45), namely $\alpha = n/(2m+1)$, with n constrained to be an even integer and $\alpha > 1$, the kinetic function $\omega(\phi)$ is rendered positive, while the kinetic function $\eta(\chi)$ becomes always negative. Thus, we can promptly find the function $\tilde{f}(\phi, \chi)$, which in this case is

$$\tilde{f}(\phi, \chi) = -\frac{1}{2}\kappa^2 \left\{ -\frac{2 \left[-b_0(1+\alpha)(t_s - \phi)^\alpha - 2\sqrt{b_0 \alpha} (t_s - \phi)^{\frac{1+\alpha}{2}} \right]}{(1+\alpha)\kappa^2} + \frac{4\sqrt{b_0 \alpha} (t_s - \chi)^{\frac{1+\alpha}{2}}}{(1+\alpha)\kappa^2} \right\}, \quad (3.75)$$

and consequently the corresponding scalar potential $V(\phi, \chi)$ reads

$$\begin{aligned} V(\phi, \chi) &= \frac{b_0 \alpha (1+\alpha)(t_s - \phi)^{-1+\alpha} + \sqrt{b_0 \alpha} (1+\alpha)(t_s - \phi)^{\frac{1}{2}(-1+\alpha)}}{(1+\alpha)\kappa^2} + \frac{\sqrt{b_0 \alpha}}{\kappa^2} (t_s - \chi)^{\frac{1}{2}(-1+\alpha)} \\ &\quad + \frac{3}{4}\kappa^2 \left\{ -\frac{2 \left[-b_0(1+\alpha)(t_s - \phi)^\alpha - 2\sqrt{b_0 \alpha} (t_s - \phi)^{\frac{1+\alpha}{2}} \right]}{(1+\alpha)\kappa^2} + \frac{4\sqrt{b_0 \alpha} (t_s - \chi)^{\frac{1+\alpha}{2}}}{(1+\alpha)\kappa^2} \right\}^2. \end{aligned} \quad (3.76)$$

Therefore, the two-field scalar-tensor theory that generates the cosmological evolution of (3.72), is given by expressions (3.74) and (3.76).

3.5 The case of multiple scalar fields

We close the current section by presenting, for completeness, the possible generalization of the above results in the case of multiple fields. Such a framework offers huge capabilities and possibilities of cosmological evolutions. We provide here the general framework of this extension following [106] (see also [128]). The scalar-tensor action in the case of n real scalar fields ϕ_i , $i = 1, \dots, n$, reads

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \sum_{i=1}^n \frac{1}{2} \omega(\phi_i) \partial_\mu \phi_i \partial^\mu \phi_i - V(\phi_i) \right\}. \quad (3.77)$$

Assuming a flat FRW background, and assuming that the scalar fields are functions only of the cosmic time t , the corresponding Friedmann equations write as

$$\sum_{i=1}^n \omega(\phi_i) \dot{\phi}^2 = -\frac{2}{\kappa^2} \dot{H}, \quad (3.78)$$

$$V(\phi_i) = \frac{1}{\kappa^2} (3H^2) - \frac{1}{2} \sum_{i=1}^n \omega(\phi_i) \dot{\phi}_i^2. \quad (3.79)$$

We choose the function $f(t)$ in order to satisfy the following equation:

$$\sum_{i=1}^n \omega_i(t) = -\frac{2}{\kappa^2} f'(t), \quad (3.80)$$

and therefore we obtain

$$\begin{aligned} \omega_i(\phi_i) &= -\frac{2}{\kappa^2} \frac{df(\phi_1, \phi_2, \dots, \phi_n)}{d\phi_i}, \\ V(\phi_1, \phi_2, \dots, \phi_n) &= \frac{1}{\kappa^2} \left[3f(\phi_1, \phi_2, \dots, \phi_n)^2 + \sum_{i=1}^n \frac{df(\phi_1, \phi_2, \dots, \phi_n)}{d\phi_i} \right], \end{aligned} \quad (3.81)$$

with the function $f(t_i) = f(t, t, \dots, t) = f(t)$. The solution of the reconstruction method is then given by

$$\phi = \chi = t, \quad H = f(t). \quad (3.82)$$

By appropriately choosing some arbitrary functions $g_n(\phi_i)$ we can make the kinetic functions $\omega_i(\phi_i)$ to be

$$\begin{aligned} \omega_1(\phi_1) &= -\frac{2}{\kappa^2} \{f'(\phi_1) + g_2(\phi_1) + \dots + g_n(\phi_1)\}, \\ \omega_2(\phi_2) &= \frac{2}{\kappa^2} g_2(\phi_2), \\ &\vdots \\ \omega_n(\phi_n) &= \frac{2}{\kappa^2} g_n(\phi_n). \end{aligned} \quad (3.83)$$

Hence, one may choose to have ghost and non-ghost scalar fields and describe the cosmological evolution in a desired way. We do not analyze this case in more details since it lies beyond the scope of the present work.

4 $F(R)$ -gravity analysis

As a last section of this work, and for completeness, we will investigate the case of $F(R)$ gravity [101, 104, 105, 129–134], which is closely related to scalar-field theory, focusing again on the Type IV realization of a Hubble rate of the form (2.2). We will follow the reconstruction procedure of [135–137] (see also [69]).

We start by recalling that in the case where $\alpha \gg 1$, the scalar-tensor potential which perfectly describes this cosmology is a hilltop-like potential of the form (2.13). We shall

assume that the parameters c_0 and b_0 appearing in (2.14) are chosen in order to satisfy $c_0 \ll 1$ and $b_0 \ll 1$, and in addition $\alpha \gg 1$. Furthermore, we remind that for an FRW metric, the Ricci scalar writes as

$$R = 6(2H^2 + \dot{H}). \quad (4.1)$$

As the cosmic time t approaches the finite time t_s , then $t - t_s \rightarrow 0$. We shall use this property later on in this section, in order to find approximate expressions for the $F(R)$ gravity. In addition, as $t \rightarrow t_s$, the Ricci scalar approaches the limiting value $R = 12c_0^2$, as can be seen by calculating its analytic form

$$R = 12c_0^2 + 24b_0c_0(-t + t_s)^\alpha + 12b_0^2(-t + t_s)^{2\alpha} - 6b_0(-t + t_s)^{-1+\alpha}\alpha. \quad (4.2)$$

We shall further investigate the physical consequences of our assumptions in the end of this section.

The Jordan frame action of a pure $F(R)$ gravity is

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} F(R), \quad (4.3)$$

with pure referring to the absence of matter fluids. Varying action (4.3) with respect to the metric tensor leads to the Friedmann equation

$$-18 \left[4H(t)^2 \dot{H}(t) + H(t) \ddot{H}(t) \right] F''(R) + 3 \left[H^2(t) + \dot{H}(t) \right] F'(R) - \frac{F(R)}{2} = 0. \quad (4.4)$$

By making use of an auxiliary scalar field ϕ (which has no kinetic term), the $F(R)$ gravity action of (4.3) can be rewritten as

$$S = \int d^4x \sqrt{-g} [P(\phi)R + Q(\phi)]. \quad (4.5)$$

By finding the exact analytic form of the functions $P(\phi)$ and $Q(\phi)$, which have an explicit dependence on the scalar field ϕ , we may easily obtain the specific $F(R)$ gravity. This can be done by varying the action of (4.5) with respect to the auxiliary scalar degree of freedom ϕ . Such a variation gives

$$P'(\phi)R + Q'(\phi) = 0, \quad (4.6)$$

with the prime now denoting differentiation with respect to ϕ . Solving Eq. (4.6) with respect to ϕ will provide us with the function $\phi(R)$, and then the $F(R)$ gravity can be easily found by substituting $\phi(R)$ to the auxiliary $F(R)$ action of (4.5). Hence, the reconstructed $F(R)$ gravity will be

$$F(\phi(R)) = P(\phi(R))R + Q(\phi(R)). \quad (4.7)$$

Let us apply the above reconstruction procedure for the case where the Hubble rate is

$$H(t) = c_0 + b_0(-t + t_s)^\alpha. \quad (4.8)$$

By varying action (4.5) with respect to the metric, we obtain the equations

$$\begin{aligned} 0 &= -6H^2 P(\phi(t)) - Q(\phi(t)) - 6H \frac{dP(\phi(t))}{dt} = 0, \\ 0 &= \left(4\dot{H} + 6H^2\right) P(\phi(t)) + Q(\phi(t)) + 2 \frac{d^2 P(\phi(t))}{dt^2} + \frac{dP(\phi(t))}{dt} = 0. \end{aligned} \quad (4.9)$$

Eliminating $Q(\phi(t))$ from (4.9) results to

$$2 \frac{d^2 P(\phi(t))}{dt^2} - 2H(t) \frac{dP(\phi(t))}{dt} + 4\dot{H} P(\phi(t)) = 0. \quad (4.10)$$

In this way, given the Hubble rate, the explicit form of $P(t)$ can be found and from this we can obtain $Q(t)$ accordingly. We mention that, due to fact that the $F(R)$ action (4.3) is mathematically equivalent to the auxiliary $F(R)$ action (4.5), the auxiliary scalar field is identical to the cosmic time t , that is $\phi = t$. This was proven in detail in the Appendix A of Ref. [135]. Substituting the Hubble rate (4.8) in (4.10), we obtain the following second-order differential equation:

$$2 \frac{d^2 P(t)}{dt^2} - 2[c_0 + b_0(-t + t_s)^\alpha] \frac{dP(t)}{dt} - 4b_0(-t + t_s)^{-1+\alpha} \alpha P(t) = 0, \quad (4.11)$$

which by setting $x = -t + t_s$ can be rewritten as

$$2 \frac{d^2 P(x)}{dx^2} - 2(c_0 + b_0 x^\alpha) \frac{dP(x)}{dx} - 2b_0 x^{-1+\alpha} \alpha P(x) = 0. \quad (4.12)$$

The general solution of the above equation is difficult to be extracted, thus we will follow approximations. In particular, since x is small the differential equation becomes

$$2 \frac{d^2 P(x)}{dx^2} - 2c_0 \frac{dP(x)}{dx} = 0, \quad (4.13)$$

with solution

$$P(x) = \frac{e^{c_0 x} c_1}{1 + c_0} + c_2, \quad (4.14)$$

with c_1, c_2 being arbitrary constants. Then, the function $Q(x)$ can easily be calculated using (4.9) as

$$Q(x) = -6c_0^2 c_2 - \frac{12c_0^2 c_1 e^{c_0 x}}{1 + c_0} - 12b_0 c_0 c_2 x^\alpha - \frac{18b_0 c_0 c_1 e^{c_0 x} x^\alpha}{1 + c_0} - 6b_0^2 c_2 x^{2\alpha} - \frac{6b_0^2 c_1 e^{c_0 x} x^{2\alpha}}{1 + c_0}. \quad (4.15)$$

Substituting the final forms of the functions $P(x)$ and $Q(x)$ into (4.6), and by solving with respect to x , we can find the function $x(R)$. For the case $t \rightarrow t_s$ that we are interested in, i.e. for $x \rightarrow 0$, we can expand the involved $P'(x)$ and $Q'(x)$. Doing so we acquire

$$P'(x) \simeq \frac{c_0 c_1}{1 + c_0} + \frac{c_0^2 c_1 x}{1 + c_0} + \frac{c_0^3 c_1 x^2}{2(1 + c_0)}, \quad (4.16)$$

and

$$\begin{aligned}
Q'(x) \simeq & x^{2\alpha} \left[-\frac{6(b_0^2 c_0 c_1)}{1+c_0} - \frac{6(b_0^2 c_0^2 c_1)x}{1+c_0} \right] + x^\alpha \left[-\frac{12(b_0 c_0^2 c_1)}{1+c_0} - \frac{12(b_0 c_0^3 c_1)x}{1+c_0} \right] \\
& + x^\alpha \left[-\frac{6(b_0 c_0^2 c_1)}{1+c_0} - \frac{6(b_0 c_0^3 c_1)x}{1+c_0} \right] + \left[-\frac{12(c_0^3 c_1)}{1+c_0} - \frac{12(c_0^4 c_1)x}{1+c_0} \right] \\
& + x^\alpha \left[-\frac{6(b_0 c_0 c_1 \alpha)}{(1+c_0)x} - \frac{6(b_0 c_0^2 c_1 \alpha)}{1+c_0} \right] \\
& + x^{2\alpha} \left\{ -\frac{12 \left[b_0^2 \left(\frac{c_1}{1+c_0} + c_2 \right) \alpha \right]}{x} - \frac{12(b_0^2 c_0 c_1 \alpha)}{1+c_0} \right\} \\
& + x^\alpha \left\{ -\frac{12 \left[b_0 c_0 \left(\frac{c_1}{1+c_0} + c_2 \right) \alpha \right]}{x} - \frac{12(b_0 c_0^2 c_1 \alpha)}{1+c_0} \right\}. \tag{4.17}
\end{aligned}$$

Since $\alpha \gg 1$, keeping the most dominant terms in (4.17) we obtain

$$Q'(x) \simeq \left(-\frac{12c_0^3 c_1}{1+c_0} - \frac{12c_0^4 c_1 x}{1+c_0} \right). \tag{4.18}$$

Hence, substituting (4.16) and (4.18) into (4.6) we acquire

$$x \simeq \frac{12c_0^3 - c_0 R + \sqrt{144c_0^6 - c_0^2 R^2}}{c_0^2 R}. \tag{4.19}$$

Thus, the $F(R)$ gravity form that can reproduce the Type IV singularity, can be obtained straightforwardly upon substitution of (4.19) into (4.7), and reads as

$$F(R) \simeq \frac{144c_0^4 c_1}{(1+c_0)R} - \frac{144c_0^5 c_1}{(1+c_0)R} + c_2 R + \frac{12c_0 c_1 \sqrt{144c_0^6 - c_0^2 R^2}}{(1+c_0)R} - \frac{12c_0^2 c_1 \sqrt{144c_0^6 - c_0^2 R^2}}{(1+c_0)R}. \tag{4.20}$$

Observing the above form, we deduce that by appropriately choosing the arbitrary constant c_2 we can make the $F(R)$ function to be the usual Einstein-Hilbert gravity plus curvature corrections. Indeed, for $c_2 = 1$, the $F(R)$ form becomes

$$F(R) \simeq R + \frac{144c_0^4 c_1}{(1+c_0)R} - \frac{144c_0^5 c_1}{(1+c_0)R} + \frac{12c_0 c_1 \sqrt{144c_0^6 - c_0^2 R^2}}{(1+c_0)R} - \frac{12c_0^2 c_1 \sqrt{144c_0^6 - c_0^2 R^2}}{(1+c_0)R}. \tag{4.21}$$

We can further simplify the final form of the $F(R)$ gravity by recalling that (4.21) is valid only near the Type IV singularity, in which case the Ricci scalar approaches the limiting value $R = R_c = 12c_0^2$ (see (4.2)). Therefore, for $R \rightarrow R_c$, the last two terms of (4.21) are nearly zero. Consequently, the final form of the $F(R)$ gravity is approximately

$$F(R) \simeq R + \frac{144c_0^4 c_1 (1-c_0)}{(1+c_0)} \frac{1}{R}. \tag{4.22}$$

This type of $F(R)$ gravity was described in [101]. Recall our initial assumptions, namely $c_0 \ll 1$ and $b_0 \ll 1$, and in addition $\alpha \gg 1$. Furthermore, as $t \rightarrow t_s$ the scalar curvature

approaches the limiting value $R \rightarrow R_c$. Therefore, we have an appealing physical picture if we additionally require that t_s is a future late-time singularity. Indeed, the $F(R)$ gravity is of the form $F(R) \sim R + \frac{C}{R}$ and the scalar curvature approaches a late-time point, which since $c_0 \ll 1$ this limiting curvature value can be chosen to have a very small value. Hence, the $F(R)$ form of (4.22) can successfully describe late-time acceleration near the future Type IV singularity, which was the goal of the present section.

We mention that the $F(R)$ gravity of the form (4.22) can consistently describe dark energy and late-time acceleration, as was demonstrated in [101], and with the presence of an R^2 term it can unify inflation with dark energy [138]. However, the model (4.22) is not completely in agreement with observations, having problems with the correct description of the gravitational interaction in the Solar System. Nevertheless, the model (4.22) turns out to be a very useful approximate form of consistent and realistic versions of $F(R)$ gravity (see [133] for a review). However, we have to note that the de Sitter solution for this model is $R_{\text{ds}} = 12c_0^2 \sqrt{-3c_1}$, thus for negative c_1 the model we found may not describe dark energy, as was shown in [139, 140], since $F''(R) < 0$. Additionally, as it was stressed in [141], in order for the scalaron mass to be less than the Planck mass, the $F''(R)$ has to be bounded for a finite R , which is not satisfied by (4.20).

5 Conclusions

In this work we demonstrated that finite time singularities of Type IV, can consistently be incorporated in the Universe's cosmological evolution, either appearing in the inflationary era, or in the late-time regime. We thoroughly discussed the most consistent framework in which these can be realized without instabilities. In particular, using only one scalar field then the linear-level instabilities can in principle occur at the cosmological time at which the phantom divide is crossed during the dynamical evolution, as we showed to happen in the case of a hilltop-like potential. On the other hand, when two fields are involved, one being canonical and one being ghost, it is possible to avoid such instabilities during the phantom-divide crossing. In addition, despite their increased level of complexity, the two-field scalar-tensor theories prove to be able to offer a plethora of possible viable cosmological scenarios, at which various types of cosmological singularities can be realized. For instance, it is possible to describe phantom inflation with the appearance of a Type IV singularity, and phantom late-time acceleration which ends in a Big Rip singularity.

For completeness, we also presented the Type IV realization in the context of $F(R)$ gravity, which is known to be connected with a scalar-field theory through conformal transformations. In particular, we showed that specifically reconstructed $F(R)$ forms can generate cosmological scenarios with a dynamical evolution corresponding to hilltop-like potentials. As we evinced, the resulting $F(R)$ gravity is of the form $R + \frac{A}{R}$, and this can be quite physically appealing if the Type IV singularity occurs at late times, since the resulting model is known to produce late-time acceleration.

In the absence of a complete understanding of the nature of cosmological singularities, we believe that with the present study we demonstrated that not all singularities can be harmful, at least for the cases where observational consistency is the main goal of a

self-consistent physical theory. Indeed, even in observable quantities that contain higher derivatives of the scale factor, the effect of singularities, and particularly of Type IV singularities, might be negligible, as for example in the jerk parameter. Definitely, a lot of effort is needed to theoretically understand the singularities, but probably these Type IV singularities might be a classical bridge between classical cosmology and the ultimate quantum gravitational cosmology, which offers a complete description or at least a remedy of these classically unwanted spacetime points. Some steps towards these directions of research are offered by loop quantum gravity, in which no singularities occur [142–146]. However, we mention that it is possible even at the classical level to remedy singularities [23] or even avoid them [60], when quantum corrections are taken into account, as in [6, 23], and thus we hope to address this issue in the future in more details.

It would be interesting to notice that phantom-non-phantom transitions as well as non-singular evolution appear also in the case of bouncing cosmology. Hence, one could follow the procedures and the techniques of the present work, in order to investigate the effects of Type IV singularities, and probably of Type III ones, in the context of various ekpyrotic [147–149] or bounce Universe scenarios [29–50]. This thorough analysis lies beyond the scope of the present work and will be addressed elsewhere.

Finally, we have to mention that in this work we did not address in general the issue of possible analyticity loss beyond $\varphi = 0$. As it is also commented by Barrow and Graham in [74], for large-field inflation potentials of the form $\sim \varphi^n$, there are fractional powers of n for which the potential is always real, and extensibility through $\varphi = 0$ can be achieved leading to a Big Crunch (for instance when the potential becomes negative). The forms of the potentials we used are exactly of this class (large-field inflation ones), or hilltop-like potentials. Nevertheless, in general, the analyticity issue around $\varphi = 0$ is very interesting, both from mathematical as well as physical point of view. This issue however deserves a detailed investigation of its own and is left for a future project.

Acknowledgments

This work is supported in part by the JSPS Grant-in-Aid for Scientific Research (S) # 22224003 and (C) # 23540296 (S.N.) and in part by MINECO (Spain), projects FIS2010-15640 and FIS2013-44881. The research of ENS is implemented within the framework of the Action “Supporting Postdoctoral Researchers” of the Operational Program “Education and Lifelong Learning” (Actions Beneficiary: General Secretariat for Research and Technology), and is co-financed by the European Social Fund (ESF) and the Greek State.

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